DM 7
Calcul matriciel

TP : A l'aide du logiciel Maxima, résoudre le système (S) suivant :

$$(S)\begin{cases} X + 2Y + 3Z - T + V = 1 \\ X + 2Y + Z - U + V = 0 \\ Y + Z + 2T - 2U = 1 \\ Y + 3Z + 2T + U + V = -1 \\ 2X - 3Y + Z + 2T + 4U + 2V = 0 \\ X + 2Y + 2Z - U + V = 1 \end{cases}$$

On pose : $A = \begin{bmatrix} 1 & 2 & 3 & -1 & 0 & 1 \\ 1 & 2 & 1 & 0 & -1 & 1 \\ 0 & 1 & 1 & 2 & -2 & 0 \\ 0 & 1 & 3 & 2 & 1 & 1 \\ 2 & -3 & 1 & 2 & 4 & 2 \\ 1 & 2 & 2 & 0 & -1 & 1 \end{bmatrix}$; $V = \begin{bmatrix} X \\ Y \\ Z \\ T \\ U \\ V \end{bmatrix}$ et $B = \begin{bmatrix} 1 \\ 0 \\ 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}$

$$(S) \Longleftrightarrow A \cdot V = B \Longleftrightarrow \underbrace{A^{-1} A}_{I} \cdot V = A^{-1} B$$

On a $A^{-1} = \begin{bmatrix} -2 & -5 & -\frac{1}{2} & -1 & \frac{1}{2} & 7 \\ 2 & 3 & 1 & 0 & 0 & -5 \\ 0 & -1 & 0 & 0 & 0 & 1 \\ -\frac{1}{2} & -3/4 & 1/8 & 0 & 1/8 & 1 \\ 1/2 & 1/4 & 1/8 & 0 & 1/8 & -1 \\ -3/2 & 5/4 & -11/8 & 1 & -3/8 & 1 \end{bmatrix}$

Donc $A^{-1} \cdot B = V$

donc $V = \begin{bmatrix} 11/2 \\ -2 \\ 1 \\ \frac{5}{8} \\ -3/6 \\ -23/8 \end{bmatrix}$

Les solutions du système S sont : ~~$X = \frac{11}{2}$; $Y = -2$; $Z = 1$; $T = \frac{5}{8}$; $U = -3/6$ et $V = \frac{-23}{8}$.~~

TP : A l'aide du logiciel Maxima, résoudre le système (S) suivant :

$$(S)\begin{cases} X + 2Y + 3Z - T + V = 1 \\ X + 2Y + Z - U + V = 0 \\ Y + Z + 2T - 2U = 1 \\ Y + 3Z + 2T + U + V = -1 \\ 2X - 3Y + Z + 2T + 4U + 2V = 0 \\ X + 2Y + 2Z - U + V = 1 \end{cases}$$

(%i7) A: matrix([1,2,3,-1,0,1], [1,2,1,0,-1,1], [0,1,1,2,-2,0], [0,1,3,2,1,1], [2,-3,1,2,4,2], [1,2,2,0,-1,1]);

$$A \quad \begin{pmatrix} 1 & 2 & 3 & -1 & 0 & 1 \\ 1 & 2 & 1 & 0 & -1 & 1 \\ 0 & 1 & 1 & 2 & -2 & 0 \\ 0 & 1 & 3 & 2 & 1 & 1 \\ 2 & -3 & 1 & 2 & 4 & 2 \\ 1 & 2 & 2 & 0 & -1 & 1 \end{pmatrix}$$

(%i11) determinant(A);

(%o11) -8

(%i8) B: matrix([1], [0], [1], [-1], [0], [1]);

$$B \quad \begin{pmatrix} 1 \\ 0 \\ 1 \\ -1 \\ 0 \\ 1 \end{pmatrix}$$

(%i14) AA: invert(A);

$$AA \quad \begin{pmatrix} 4 & 4 & \frac{5}{2} & -1 & \frac{1}{2} & -8 \\ 2 & 3 & 1 & 0 & 0 & -5 \\ 0 & -1 & 0 & 0 & 0 & 1 \\ 1 & \frac{3}{2} & \frac{7}{8} & 0 & \frac{1}{8} & -\left(\frac{11}{4}\right) \\ 2 & \frac{5}{2} & \frac{7}{8} & 0 & \frac{1}{8} & -\left(\frac{19}{4}\right) \\ -6 & -\left(\frac{11}{2}\right) & -\left(\frac{29}{8}\right) & 1 & -\left(\frac{3}{8}\right) & \frac{49}{4} \end{pmatrix}$$

(%i15) X: AA.B;

$$X \quad \begin{pmatrix} -\left(\frac{1}{2}\right) \\ -2 \\ 1 \\ -\left(\frac{7}{8}\right) \\ -\left(\frac{15}{8}\right) \\ \frac{13}{8} \end{pmatrix}$$

(%i1) solve([x+2·y+3·z−t+v= 1, x+2·y+z−u−v=0,y+z+2·t−2·u=1, y+3·z+2·t+u+v=−1,2·x−3·y+z+2·t+4·u+2·v=0,x+2·y+2·z−u+v=1],[x,y,z,t,u,v]) ;

(%o1) $\left[\left[x=\frac{7}{12}, y=-\left(\frac{19}{16}\right), z=\frac{35}{48}, t=-\left(\frac{15}{32}\right), u=-\left(\frac{115}{96}\right), v=\frac{13}{96}\right]\right]$

Exercice 6 :

$$(S) = \begin{cases} I_1 - I_2 - I_3 = 0 \\ 10 I_1 \quad + 6 I_3 = 12 \\ 10 I_1 + 2 I_2 \quad = 18 \end{cases} \qquad R = \begin{bmatrix} 1 & -1 & -1 \\ 10 & 0 & 6 \\ 10 & 2 & 0 \end{bmatrix}$$

1) Calcul du déterminant :

$$|R| = \left(0 + (-20) + (-60)\right) - \left(0 + 12 + 0\right) = -80 - 12 = \boxed{-92}$$

Donc R est inversible car $R \neq 0$

2) $$R = \begin{bmatrix} 1 & -1 & -1 \\ 10 & 0 & 6 \\ 10 & 2 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 12 \\ 18 \end{bmatrix} \qquad X = \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix}$$

$$(S) \iff R \cdot X = B \iff \underbrace{R^{-1} R}_{} X = R^{-1} B \iff X = R^{-1} \cdot B$$

$$I X = R^{-1} B$$

$$A^{-1} = \frac{1}{|A|}\ {}^{t}\mathrm{Co}A$$

$$\mathrm{Co}A = \begin{pmatrix} +\begin{vmatrix} 0 & 6 \\ 2 & 0 \end{vmatrix} & -\begin{vmatrix} 10 & 6 \\ 10 & 0 \end{vmatrix} & +\begin{vmatrix} 10 & 0 \\ 10 & 2 \end{vmatrix} \\ -\begin{vmatrix} -1 & -1 \\ 2 & 0 \end{vmatrix} & +\begin{vmatrix} 1 & -1 \\ 10 & 0 \end{vmatrix} & -\begin{vmatrix} 1 & -1 \\ 10 & 2 \end{vmatrix} \\ +\begin{vmatrix} -1 & -1 \\ 0 & 6 \end{vmatrix} & -\begin{vmatrix} 1 & -1 \\ 10 & 6 \end{vmatrix} & \begin{vmatrix} 1 & -1 \\ 10 & 0 \end{vmatrix} \end{pmatrix}$$

$$\mathrm{Co}A = \begin{pmatrix} +(0\times 0 - 2\times 6) & -(10\times 0 - 10\times 6) & +(10\times 2 - 10\times 0) \\ -((-1)\times 0 - 2\times(-1)) & +(1\times 0 - 10\times(-1)) & -(1\times 2 - 10\times(-1)) \\ +((-1)\times 6 - 0\times(-1)) & -(1\times 6 - 10\times(-1)) & +(1\times 0 - 10\cdot(-1)) \end{pmatrix}$$

$$\mathrm{Co}A = \begin{pmatrix} -12 & 60 & 20 \\ -2 & 10 & -12 \\ -6 & -16 & 10 \end{pmatrix} \Leftrightarrow {}^{t}\mathrm{Co}A = \begin{pmatrix} -12 & -2 & -6 \\ 60 & 10 & -16 \\ 20 & -12 & 40 \end{pmatrix}$$

Donc $A^{-1} = \frac{1}{|A|} \times {}^{t}\mathrm{Co}A = \frac{1}{-92}\begin{pmatrix} -12 & -2 & -6 \\ 60 & 10 & -16 \\ 20 & -12 & 10 \end{pmatrix}$

$$A^{-1} = \frac{1}{92}\begin{pmatrix} 12 & 2 & 6 \\ -60 & -10 & 16 \\ -20 & 12 & -10 \end{pmatrix}$$

$$X = A^{-1} \times B$$

$$X = R^{-1} \times B = \frac{1}{92} \times \begin{pmatrix} 12 & 2 & 6 \\ -60 & -10 & 16 \\ -20 & 12 & -10 \end{pmatrix} \times \begin{pmatrix} 0 \\ 12 \\ 18 \end{pmatrix}$$

$$= \frac{1}{92} \begin{pmatrix} (0 + 24 + 108) \\ (-\cancel{720} + (-120) + 288) \\ (-\cancel{360} + 144 + (-180)) \end{pmatrix}$$

$$X = \frac{1}{92} \begin{pmatrix} 132 \\ -\cancel{560}\ 168 \\ -\cancel{324}\ -36 \end{pmatrix} \simeq \begin{pmatrix} 1{,}4 \\ \cancel{-6} \\ \cancel{-3{,}5} \end{pmatrix}$$

Cc

$$I_1 = \frac{132}{92} = \frac{33}{23} \simeq 1{,}4\ A$$

$$I_2 = \frac{168}{92} = \frac{42}{23} \simeq 1{,}8\ A$$

$$I_3 = -\frac{9}{23} \simeq -0{,}4\ A$$

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(%i1) R: matrix(
 [1,-1,-1],
 [10,0,6],
 [10,2,0]
);
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$$R \quad \begin{pmatrix} 1 & -1 & -1 \\ 10 & 0 & 6 \\ 10 & 2 & 0 \end{pmatrix}$$

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(%i3) determinant(R);
(%o3) -92
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(%i3) B: matrix(
 [0],
 [12],
 [18]
);
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$$B \quad \begin{pmatrix} 0 \\ 12 \\ 18 \end{pmatrix}$$

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(%i4) R_inv : invert(R);
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$$R\_inv \quad \begin{pmatrix} \frac{3}{23} & \frac{1}{46} & \frac{3}{46} \\ -\left(\frac{15}{23}\right) & -\left(\frac{5}{46}\right) & \frac{4}{23} \\ -\left(\frac{5}{23}\right) & \frac{3}{23} & -\left(\frac{5}{46}\right) \end{pmatrix}$$

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(%i5) I : R_inv . B;
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$$I \quad \begin{pmatrix} \frac{33}{23} \\ \frac{42}{23} \\ -\left(\frac{9}{23}\right) \end{pmatrix}$$

Exercice 7

$\det(M) = 4 \neq 0$ donc inversible

$\hookrightarrow$ $M^{-1} \begin{pmatrix} -2 & 9/2 & -6 \\ -1 & \frac{5}{2} & -4 \\ -\frac{1}{2} & 1 & -3/2 \end{pmatrix}$

1) $S = \begin{pmatrix} 2 \\ -4 \\ 6 \end{pmatrix} \Rightarrow M^{-1} \cdot S = \begin{pmatrix} -58 \\ -36 \\ -14 \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$$S: \begin{cases} 2x + y - z = 4 \\ x - 2y + z = -1 \\ -x + 2y + 2z = -5 \end{cases}$$

Donc $A = \begin{pmatrix} 2 & 1 & -1 \\ 1 & -2 & 1 \\ -1 & 2 & 2 \end{pmatrix}$ $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ $B = \begin{pmatrix} 4 \\ -1 \\ -5 \end{pmatrix}$

D'où $A \cdot X = B$

Soit $A^{-1} = \frac{1}{|A|} \cdot {}^tCoA$ pour $|A| \neq 0$

$$|A| = \left(2 \times (-2) \times 2 + 1 \times 2 \times (-1) + 1 \times 1 \times (-1)\right) - \left((-1) \times (-2) \times (-1) + 2 \times 1 \times 2 + 2 \times 1 \times 2\right)$$
$$= \left(-8 + (-1) + (-1)\right) - \left(-1 + 2 + 4\right)$$
$$= (-10 - 5)$$
$$|A| = (-15)$$

Donc $|A| \neq 0$.

$A^{-1} = \frac{1}{|A|} \cdot {}^tCoA$ $\Rightarrow$ $CoA = \begin{pmatrix} + \begin{vmatrix} -2 & 1 \\ 2 & 2 \end{vmatrix} & - \begin{vmatrix} 1 & 1 \\ -1 & 2 \end{vmatrix} & + \begin{vmatrix} 1 & -2 \\ -1 & 2 \end{vmatrix} \\ - \begin{vmatrix} 1 & -1 \\ 2 & 2 \end{vmatrix} & + \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} & - \begin{vmatrix} 2 & 1 \\ -1 & 2 \end{vmatrix} \\ + \begin{vmatrix} 1 & -1 \\ -2 & 1 \end{vmatrix} & - \begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix} & + \begin{vmatrix} 2 & 1 \\ 1 & -2 \end{vmatrix} \end{pmatrix}$

$$coA = \begin{pmatrix} +((-2) \times 2 - 2 \times 1) & -(1 \times 2 - (-1) \times 1) & +(1 \times 2 - (-1) \times (-2)) \\ -(1 \times 2 - 2 \times (-1)) & +(2 \times 2 - (-1) \times (-1)) & -(2 \times 2 - (-1) \times 1) \\ +(1 \times 1 - (-2) \times (-1)) & -(2 \times 1 - 1 \times (-1)) & +(2 \times (-2) - 1 \times 1) \end{pmatrix}$$

$$CoA = \begin{pmatrix} -6 & -3 & 0 \\ -4 & 3 & -5 \\ 1 & -3 & 5 \end{pmatrix}$$

$${}^tCoA = \begin{pmatrix} -6 & -4 & 1 \\ -3 & 3 & -3 \\ 0 & -5 & 5 \end{pmatrix}$$

Donc $A^{-1} = \frac{1}{|A|} \, {}^tCoA$

$$A^{-1} = \frac{1}{-15} \begin{pmatrix} -6 & -4 & -1 \\ -3 & 3 & -3 \\ 0 & 5 & 5 \end{pmatrix}$$

$$A^{-1} = \frac{1}{15} \begin{pmatrix} 6 & 4 & -1 \\ 3 & -3 & 3 \\ 0 & 5 & 5 \end{pmatrix}$$

Soit $A^{-1} A \cdot X = A^{-1} B$

$I_3 \cdot X = A^{-1} B$ $\Rightarrow X = A^{-1} \cdot B$.

$$A^{-1} \cdot B = \frac{1}{15} \begin{pmatrix} 6 & 4 & -1 \\ 3 & -3 & 3 \\ 0 & 5 & 5 \end{pmatrix} \times \begin{pmatrix} 4 \\ -1 \\ -5 \end{pmatrix}$$

$$= \frac{1}{15} \begin{pmatrix} 24 - 4 - 5 \\ 12 + 3 - 15 \\ 0 - 5 - 25 \end{pmatrix}$$

$$= \frac{1}{15} \times \begin{pmatrix} 15 \\ 0 \\ -30 \end{pmatrix}$$

$$X = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}$$

Question à 100 balles : A + on $(A \times B)^{-1} = A^{-1} B^{-1}$ Non car : $(AB)^{-1} = B^{-1} A^{-1}$

en effet, $(AB)^{-1} \times (AB) = I$ par définition

or : $B^{-1} \underbrace{A^{-1} \times A}_{I} B$

$= B^{-1} \times B = I$ ok

Donc $(AB)^{-1} = B^{-1} \times A^{-1}$