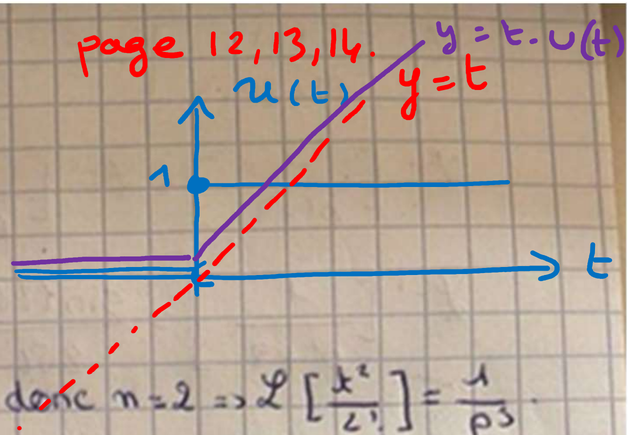
Corrigé du DM12 et du DM13
1ALT

① III. Transformation de Laplace inverse.

page 12, 13, 14.



$\mathcal{L}^{-1}[1/p] = u(t)$ avec $u(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$

Notation abusive.

donc $\mathcal{L}^{-1}[1/p] = 1, (t \geq 0)$ ou $1 \cdot U(t)$

$\mathcal{L}^{-1}[2/p^3] = t^2$ $\qquad \mathcal{L}^{-1}\left[\frac{1}{p^3}\right] = \frac{t^2}{2}$

↳ $\mathcal{L}\left[\frac{t^n}{n!}\right] = \frac{1}{p^{n+1}}$, ici nous identifions $n+1 = 3$ donc $n = 2 \Rightarrow \mathcal{L}\left[\frac{t^2}{2!}\right] = \frac{1}{p^3}$.

↳ On multiplie tout par 2 : $\mathcal{L}[t^2] = \frac{2}{p^3}$ donc $\mathcal{L}^{-1}[2/p^3] = t^2$.

$$\mathcal{L}^{-1}\left[\frac{n!}{p^{n+1}}\right] = t^n$$

$$n! \, \mathcal{L}^{-1}\left[\frac{1}{p^{n+1}}\right] = t^n$$

$\mathcal{L}^{-1}[1/p+3] = e^{-3t}, t \geq 0$

↳ $\mathcal{L}[e^{-at}] = 1/p+a$. avec ici a vaut 3.

$\mathcal{L}^{-1}[p/p^2+4] = \cos(2t), t \geq 0$

↳ $\mathcal{L}[\cos(\omega t)] = p/p^2+\omega^2$. avec $\omega = 2$.

$\mathcal{L}^{-1}[2/p^2+4] = \sin(2t), t \geq 0$

↳ $\mathcal{L}[\sin(\omega t)] = \omega/p^2+\omega^2$ avec $\omega = 2$.

$\mathcal{L}^{-1}[1/(p+3)^2] = te^{-3t}, t \geq 0$

↳ $\mathcal{L}[e^{-at}t^n] = n!/(p+a)^{n+1}$ avec $n = 1$ et $a = 3$.

$\frac{1}{3!}$ $\mathcal{L}^{-1}[3! \cdot 1/p^4]$ à pour formule $\mathcal{L}[t^n/n!] = 1/p^{n+1}$ soit ici, $\mathcal{L}[\frac{t^3}{3!}] = 1/p^4$

↳ $\mathcal{L}^{-1}[1/p^4] = \frac{t^3}{3!} = \frac{t^3}{6}$, $t \geq 0$.

$\mathcal{L}^{-1}\left[\frac{2}{p} + \frac{3}{p^3} - \frac{1}{p^5}\right] = 2 + 3/2\, t^2 + -1/24\, t^4$, $t \geq 0$.

↳ Décomposant : $\mathcal{L}^{-1}[1/p] = 1$ donc $\mathcal{L}^{-1}[2/p] = 2$.

$\mathcal{L}^{-1}[3/p^3] = 3 \times t^2/2 = 3/2\, t^2$.

$\mathcal{L}^{-1}[-1/p^5] = \frac{-t^4}{4!} = \frac{-t^4}{24}$

$\mathcal{L}^{-1}\left[\frac{5p}{p^2+9}\right] = 5\cos(3t)$, $t \geq 0$

↳ Par identification : $\mathcal{L}[\cos(\omega t)] = \frac{p}{p^2+\omega^2}$ avec $\omega = 3$.

② Exercice.

$$\mathcal{L}\left[x^3 - 5x + 1\right] = \frac{3!}{p^4} - 5 \times \frac{1}{p^2} + \frac{1}{p} = \frac{6}{p^4} - \frac{5}{p^2} + \frac{1}{p}$$

$$\mathcal{L}\left[\sin(3t) . u(t)\right] = \frac{3}{p^2 + 9}$$

$$\mathcal{L}\left[e^{-2t} . \cos(5t) . u(t)\right] = \frac{p+2}{(p+2)^2 + 5^2}$$

$$\mathcal{L}^{-1}\left[\frac{1}{(p+7)^5}\right] = \mathcal{L}^{-1}\left[\frac{1}{(p+a)^{n+1}}\right] = \frac{t^n}{n!} e^{-at} \text{ avec } n = 4 \text{ et } a = 7.$$

$$\mathcal{L}^{-1}\left[\frac{1}{(p+7)^5}\right] = \frac{t^4}{24}\, e^{-7t}, \; t \geq 0.$$

$$\mathcal{L}^{-1}\left[\frac{p+1}{p^2+2p+5}\right] = \mathcal{L}^{-1}\left[\frac{p+1}{(p+1)^2+2^2}\right] = \cos(2t)\, e^{-t}, \; t \geq 0.$$

$$\mathcal{L}^{-1}\left[\frac{1}{p^2+8}\right] = \mathcal{L}^{-1}\left[\frac{1}{p^2+a^2}\right] = \frac{\sin(at)}{a} \quad \text{avec } a = \sqrt{8} = 2\sqrt{2}.$$

$$\mathcal{L}^{-1}\left[\frac{1}{p^2+8}\right] = \frac{\sin(2\sqrt{2}t)}{2\sqrt{2}}.$$

* $\frac{1}{3}\mathcal{L}^{-1}[\frac{3\times 1}{p^2+9}] = \frac{1}{3}\sin(3t)$

$$\mathcal{L}^{-1}\left[\frac{\omega}{p^2+\omega^2}\right] = \sin\omega t$$

* $\mathcal{L}^{-1}[\frac{2p+6}{(p+3)^2}] = \mathcal{L}^{-1}\left[\frac{2(p+3)}{(p+3)^2}\right] = \mathcal{L}^{-1}\left[\frac{2}{p+3}\right]$

$$\mathcal{L}^{-1}\left[\frac{1}{p+a}\right] = e^{-at}$$

$$\mathcal{L}^{-1}\left[\frac{2p+6}{(p+3)^2}\right] = 2.e^{-3t}$$

* $F(p) = \frac{3p+1}{p^2-9} = \frac{3p+1}{(p+3)(p-3)} = \frac{a}{p-3} + \frac{b}{p+3} = \frac{5}{3}\times\frac{1}{p-3} + \frac{4}{3}\times\frac{1}{p+3}$

$$\begin{cases} a = \left[(p-3).F(p)\right]_{p=3} = \left[\frac{3p+1}{p+3}\right]_{p=3} = \frac{10}{6} = \frac{5}{3} \\ b = \left[(p+3).F(p)\right]_{p=-3} = \left[\frac{3p+1}{p-3}\right]_{p=-3} = \frac{-8}{-6} = \frac{4}{3} \end{cases}$$

* $\mathcal{L}^{-1}\left[\frac{2p+6}{(p+3)^5}\right] = \mathcal{L}^{-1}\left[\frac{2(p+3)}{(p+3)^5}\right] = \frac{2}{3!}\mathcal{L}^{-1}\left[\frac{3!\times 1}{(p+3)^4}\right] = \frac{1}{3}t^3.e^{-3t}$

$$\boxed{\mathcal{L}^{-1}\left[\frac{3p+1}{p^2-9}\right] = \frac{5}{3}.e^{3t} + \frac{4}{3}e^{-3t}}$$

$$\mathcal{L}^{-1}\left[\frac{n!}{(p+a)^{n+1}}\right] = t^n.e^{-at}$$