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Exercice 3 A l'aide d'un changement de variable, calculer les intégrales suivantes :

$$I = \int_0^{\frac{\pi}{2}} \frac{\cos(t)}{(1+\sin(t))^2}\,dt \quad ; \quad K = \int_1^{e} \frac{\ln^3(t) + \ln^2(t) + 1}{t}\,dt \quad ;$$

$$L = \int_0^1 \frac{dx}{(x^2+1)^2} \text{ (on posera x=tan(t)).}$$

$$I = \int_0^{\frac{\pi}{2}} \frac{\cancel{\cos(t)}}{(1+\sin(t))^2}\,dt$$

Méthode 1

On pose $x = \sin t$ est bijective sur $[0, \pi/2]$.

Bornes : $\begin{cases} t = 0 \\ t = \frac{\pi}{2} \end{cases} \Longleftrightarrow \begin{cases} x = \sin 0 = \boxed{0} \\ x = \sin\frac{\pi}{2} = \boxed{1} \end{cases}$

relation entre dx et dt : $x = \sin t$

"dérivée de x par rapport à t" $\leftarrow \frac{dx}{dt} = (\sin t)' = \cos t \Longleftrightarrow \boxed{dx = \cos t \cdot dt}$

$$I = \int_0^1 \frac{dx}{(\underbrace{1+x}_{U})^2} = \left[-\frac{1}{1+x}\right]_0^1 = -\frac{1}{2} + 1 = \frac{1}{2} \qquad \int \frac{U'}{U^2}\,dx = -\frac{1}{U} + cte$$

$U \Rightarrow U' = 1$

Méth 2 $x = \sin t + 1$. Bornes : $\begin{cases} x = 1 \\ x = 2 \end{cases}$ $\frac{dx}{dt} = \cos t \Longleftrightarrow dx = \cos t\, dt$

$$I = \int_1^2 \frac{dx}{x^2} = \left[-\frac{1}{x}\right]_1^2 = -\frac{1}{2} + 1 = \frac{1}{2}$$

$$K = \int_1^e \frac{\ln^3(t) + \ln^2(t) + 1}{t}\,dt$$

On pose $\boxed{x = \ln t}$ est bijective $[1;e]$.

Bornes : $\begin{cases} t = 1 \\ t = e \end{cases} \iff \begin{cases} x = \ln 1 = \boxed{0} \\ x = \ln 2 = \boxed{1} \end{cases}$

relation entre dx et dt : $x = \ln t$

$$\frac{dx}{dt} = (\ln t)' = \frac{1}{t}$$

$$\boxed{dx = \frac{dt}{t}}$$

$$K = \int_0^1 (x^3 + x^2 + 1)\,dx = \left[\frac{x^4}{4} + \frac{x^3}{3} + x\right]_0^1 = \frac{1}{4} + \frac{1}{3} + 1 = \frac{3+4+12}{12}$$

$$= \frac{19}{12}$$

$$L = \int_0^1 \frac{dx}{(x^2+1)^2}$$ (on posera x=tan(t)).

$\boxed{x = \tan t} \Longleftrightarrow t = \arctan x$

↳ est bijective sur $]0;\pi/2[$.

$$\frac{dx}{dt} = (\tan t)'$$

Bornes : $\begin{cases} x = 0 \\ x = 1 \end{cases} \Longleftrightarrow \begin{cases} t = \arctan 0 = \boxed{0} \\ t = \arctan 1 = \boxed{\frac{\pi}{4}} \end{cases}$

$$\frac{dx}{dt} = (1+\tan^2 t)$$

$$\boxed{dx = (1+\tan^2 t)\,dt}$$

$$L = \int_0^{\pi/4}$$

$$1+\tan^2 t = 1 + \frac{\sin^2 t}{\cos^2 t} = \frac{\cos^2 t + \sin^2 t}{\cos^2 t}$$

$$\boxed{1+\tan^2 t = \frac{1}{\cos^2 t}}$$

$$L = \int_0^{\pi/4} \frac{(1+\tan^2 t)\,dt}{(\tan^2 t+1)^2} = \int_0^{\pi/4} \frac{1}{\tan^2 t+1}\,dt$$

$$L = \int_0^{\pi/4} \underbrace{\cos^2 t}_{\frac{1+\cos(2t)}{2}}\,dt = \frac{1}{2}\int_0^{\pi/4} (1+\cos 2t)\,dt = \frac{1}{2}\left[t + \frac{\sin(2t)}{2}\right]_0^{\pi/4} = \frac{1}{2}\left(\frac{\pi}{4} + \frac{\overbrace{\sin\pi/2}^{1}}{2}\right)$$

$$L = \frac{1}{8}(\pi+2)$$