CHAPITRE 4 – Exercice 4 page 23

Exercice 4 Calculer les intégrales suivantes :

paire × impaire = impaire

$$I = \int_{-7\pi}^{7\pi} \underbrace{\cos^2(x)}_{} \underbrace{\sin^3(5x)}_{} .dx = 0 \ ; \ J = \int_1^2 \frac{x^2}{\sqrt{x}} dx \ ; \ K(x) = \int (1+\tan^2(x)).\tan^3(x).dx \ ;$$

[−7π ; 7π] centrée en 0

$$L(x) = \int \frac{1}{x^2}.e^{\frac{1}{x}}.dx \ ; \ M(x) = \int x^2 e^{3x}.dx \ ; \ N = \int_0^1 \frac{dt}{e^{-t}+1} \ ; \ P = \int_1^3 \frac{x}{x^4+1} dx \ ;$$

$$Q = \int_0^{\frac{\pi}{2}} \cos(x).\sin^5(x).dx \ ; \ R = \int_{-1}^0 \frac{dx}{\sqrt{e^{3x}}} \ ; \ T(x) = \int \frac{dx}{x^2-6x+13} \ ;$$

$$U(x) = \int \frac{x^4}{x^4+2x^3-2x-1} dx \ ; \ V(t) = \int (t^3+2t+3).\ln(t).dt$$

$$J = \int_1^2 \frac{x^2}{\sqrt{x}}\,dx$$

$$\frac{x^2}{\sqrt{x}} = \frac{x^2}{x^{\frac{1}{2}}} = x^{2-\frac{1}{2}} = x^{\frac{3}{2}}$$

$$J = \int_1^2 x^{3/2}\,dx = \left[\frac{x^{3/2+1}}{\underbrace{3/2+1}_{5/2}}\right]_1^2 = \frac{2}{5}\left(2^{5/2} - 1^{5/2}\right)$$

$$\int x^{\alpha}\,dx = \frac{x^{\alpha+1}}{\alpha+1} + cte \qquad \alpha \neq -1$$

$$2^{\frac{5}{2}} = 2^{\frac{4}{2}+\frac{1}{2}} = 2^{2+\frac{1}{2}} = 2^2 \cdot 2^{1/2} = 4\sqrt{2}$$

$$J = \frac{2}{5}\left(4\sqrt{2} - 1\right)$$

$K(x) = \int (1+\tan^2(x)).\tan^3(x).dx$;

Méth 1 $t = \tan x$

$\frac{dt}{dx} = (\tan x)' = 1 + \tan^2 x$

$\Leftrightarrow dt = (1+\tan^2 x)\,dx$

$\cancel{K(x)} = \int t^3\, dt = \frac{t^4}{4} + cte$

$t = \tan x$

$K(x) = \frac{\tan^4 x}{4} + cte$

$x \neq \frac{\pi}{2} + k\pi \; ; k \in \mathbb{R}$

Méth 2 $K(x) = \int \underbrace{(1+\tan^2 x)}_{}.\tan^3 x\, dx$

$\int U'.U^3\,dx = \frac{U^4}{4} + cte$

$K(x) = \frac{\tan^4 x}{4} + cte$

$$L(x) = \int \frac{1}{x^2} . e^{\frac{1}{x}} . dx$$

Méth 1

$t = \frac{1}{x}$

$\frac{dt}{dx} = \left(\frac{1}{x}\right)' = -\frac{1}{x^2}$

$\Leftrightarrow \quad dt = -\frac{1}{x^2} dx$

$-\int e^t \, dt = -e^t + cte$

$L(x) = -e^{1/x} + cte$

Méth 2

$L(x) = -\int -\frac{1}{x^2} e^{1/x} dx \qquad 1/x = u \Rightarrow u' = -\frac{1}{x^2}$

$\int u' . e^u dx = e^u + cte$

$L(x) = -e^{1/x} + cte$

$M(x) = \int \overset{P}{x^2}\overset{E}{e^{3x}}.dx$ $\boxed{IPP}$ ALPES — Cos, Sin, Ch, Sh. Arccos Arcsin ln poly. exp

$\int v' e^{v} dx$

IPP1 $\begin{cases} u = x^2 \\ v' = e^{3x} \end{cases} \Rightarrow \begin{cases} u' = 2x \\ v = \frac{e^{3x}}{3} \end{cases}$

$\int u v' \, dx = uv - \int u' v \, dx$

$M(x) = \frac{x^2 e^{3x}}{3} - \int \frac{2x e^{3x}}{3} dx = \frac{x^2 e^{3x}}{3} - \frac{2}{3} \underbrace{\int x e^{3x} dx}_{\frac{x e^{3x}}{3} - \int \frac{e^{3x}}{3} dx}$

IPP2 $\begin{cases} u = x \\ v' = e^{3x} \end{cases} \Rightarrow \begin{cases} u' = 1 \\ v = \frac{e^{3x}}{3} \end{cases}$

$$M(x) = \int x^2 e^{3x}.dx = \frac{x^2 e^{3x}}{3} - \frac{2}{3}\left[\frac{x e^{3x}}{3} - \frac{1}{3}\underbrace{\int e^{3x} dx}_{\frac{e^{3x}}{3}}\right]$$

$$= \frac{x^2 e^{3x}}{3} - \frac{2}{3}\left(\frac{x e^{3x}}{3} - \frac{e^{3x}}{9}\right) + cte$$

$$= e^{3x}\left(\frac{x^2}{3} - \frac{2}{9}x + \frac{2}{27}\right) + cte$$

$$M(x) = \frac{e^{3x}}{27}\left(9x^2 - 6x + 2\right) + cte$$

$$N = \int_0^1 \frac{dt}{e^{-t}+1}$$

Méth 1 : $x = e^{-t}$ ou $x = e^{-t}+1$ ou $x = e^{t}$

$$\boxed{x = e^{-t}+1}$$

Bornes : $t = 0 \iff x = e^{0}+1 = 2$

$t = 1 \iff x = e^{-1}+1$

dx et dt : $x = e^{-t}+1$

$$\frac{dx}{dt} = \left(e^{-t}+1\right)' = -e^{-t}$$

$$\iff \boxed{dx = -e^{-t}\,dt}$$

$$N = \int_2^{e^{-1}+1} - \frac{dx}{x(x-1)}$$

$$-\int_a^b f(x)\,dx = \int_b^a f(x)\,dx$$

$$dt = -\frac{dx}{e^{-t}} \iff dt = \frac{-dx}{x-1}$$

$$e^{-t} = x-1$$

$$N = \int_{e^{-1}+1}^{2} \frac{dx}{x(x-1)} = \int_{e^{-1}+1}^{2} \left(\frac{-1}{x} + \frac{1}{x-1}\right) dx$$

$$f(x) = \frac{1}{x(x-1)} = \frac{a}{x} + \frac{b}{x-1}$$

$$a = \left[x\,f(x)\right]_{x=0} = -1$$

$$b = \left[(x-1)\,f(x)\right]_{x=1} = 1$$

$$N = \int_0^1 \frac{dt}{e^{-t}+1} = \int_{e^{-1}+1}^{2} \left(\frac{1}{x-1} - \frac{1}{x}\right) dx$$

$$N = \Big[\ln|x-1| - \ln|x| \Big]_{e^{-1}+1}^{2}$$

$$= \underbrace{\ln(1)}_{0} - \ln 2 - \left(\ln\left(e^{-1}\right) - \ln\left(e^{-1}+1\right)\right)$$

$$N = \underbrace{-\ln 2 + \ln e} + \ln\left(\frac{1}{e}+1\right)$$

$$N = \ln\left(\frac{e}{2}\right) + \ln\left(\frac{1+e}{e}\right)$$

$$\ln A - \ln B = \ln\left(\frac{A}{B}\right)$$

$$\ln A + \ln B = \ln(AB)$$

$$N = \ln\left(\frac{e}{2} \times \frac{1+e}{e}\right)$$

$$N = \ln\left(\frac{1+e}{2}\right)$$

Méth 2 : $N = \int_0^1 \frac{dt}{e^{-t}+1} \times \frac{e^t}{e^t}$

$$e^{-t} = \frac{1}{e^t}$$

$$N = \int_0^1 \frac{e^t dt}{1+e^t} \qquad \int \frac{U'}{U} dt = \ln|U| + cte$$

$$U \Rightarrow U' = e^t$$

$$N = \Big[\ln|1+e^t| \Big]_0^1$$

$$N = \ln(1+e) - \ln(2) = \ln\left(\frac{1+e}{2}\right)$$

$$P = \int_1^3 \frac{x}{x^4+1}\,dx$$

$t = \cancel{x}\ x^2$

Méth 1 $\boxed{t = x^2}$

Borns : $\begin{cases} x = 1 \\ x = 3 \end{cases} \Rightarrow \begin{cases} t = 1 \\ t = 9 \end{cases}$

$\frac{dt}{dx} = 2x \iff \cancel{dt = 2x\,dx}$ $\boxed{x\,dx = \frac{dt}{2}}$

$P = \int_1^9 \frac{\frac{dt}{2}}{t^2+1} = \frac{1}{2}\int_1^9 \frac{dt}{t^2+1}$

$P = \frac{1}{2}\left[\arctan(t)\right]_1^9$

$P = \frac{1}{2}\arctan(9) - \underbrace{\arctan(1)}_{\pi/4}$

Méth 2

$P = \int_1^3 \frac{x}{x^4+1}\,dx = \frac{1}{2}\int_1^3 \frac{2x}{(x^2)^2+1}\,dx$

$\int \frac{U'}{U^2+1}\,dx = \mathrm{Arctan}\,U + cte$

ici $U = x^2 \Rightarrow U' = 2x$

$P = \frac{1}{2}\left[\arctan(x^2)\right]_1^3 = \dots$

$$Q = \int_0^{\frac{\pi}{2}} \cos(x).\sin^5(x).dx$$

Méth 1 $t = \sin x$

Borns : $\begin{cases} x = 0 \iff t = \sin 0 = 0 \\ x = \pi/2 \iff t = \sin\frac{\pi}{2} = 1 \end{cases}$

$$\frac{dt}{dx} = (\sin x)' = \cos x$$

$$dt = \cos x \, dx$$

$$Q = \int_0^1 t^5 \, dt = \left[\frac{t^6}{6}\right]_0^1 = \frac{1}{6}$$

Méth 2

$$Q = \int_0^{\pi/2} \cos x . \sin^5 x dx$$

$$\int U' . U^5 dx = \frac{U^6}{6} + cte$$

$$Q = \left[\frac{\sin^6 x}{6}\right]_0^{\pi/2} = \frac{\sin^6 \pi/2}{6} = \frac{1}{6}$$

$$R = \int_{-1}^{0} \frac{dx}{\sqrt{e^{3x}}}$$

$$\frac{1}{\sqrt{e^{3x}}} = \frac{1}{\left(e^{3x}\right)^{1/2}} = \frac{1}{e^{\frac{3x}{2}}} = e^{-\frac{3x}{2}}$$

$$-\frac{3x}{2} = u \Rightarrow u' = -\frac{3}{2}$$

$$R = \frac{-2}{3}\int_{-1}^{0} \frac{-3}{2} \times e^{-\frac{3x}{2}}\, dx$$

$$\int u' e^{u}\, dx = e^{u} + cte$$

$$R = -\frac{2}{3}\left[e^{-\frac{3x}{2}}\right]_{-1}^{0} = -\frac{2}{3}\left(e^{0} - e^{3/2}\right) = \frac{2}{3} e^{3/2}$$

$$3/2 = 1 + \frac{1}{2}$$

$$R = \frac{2}{3} e\sqrt{e}$$

$$T(x)=\int\frac{dx}{x^2-6x+13}$$

$\int \frac{u'}{u}\,dx = \ln|u| + cte$

DSE

$$\boxed{\int \frac{u'}{u^2+1}\,dx = \arctan u + cte}$$

$f(x) = \frac{1}{x^2-6x+13} =$

$\to \Delta = 36-4\times 13 < 0$

$x^2-6x+\overbrace{13}^{9+4} = (x-3)^2+4$

$(x-3)^2+4$

$= 4\left[\frac{(x-3)^2}{4}+1\right]$

$T(x) = \frac{1}{4}\times 2\int \frac{\frac{1}{2}\times dx}{\left(\frac{x-3}{2}\right)^2+1} = \frac{1}{2}\times \arctan\left(\frac{x-3}{2}\right)+cte$

$\frac{x-3}{2} = u \Rightarrow u' = \frac{1}{2}(x-3)' = \frac{1}{2}$

$$V(t) = \int \underbrace{(t^3 + 2t + 3)}_{P}.\underbrace{\ln(t)}_{L}.dt$$ IPP : ALPES.

$$\begin{cases} u = \ln t \\ v' = t^3 + 2t + 3 \end{cases} \Rightarrow \begin{cases} u' = \dfrac{1}{t} \\ v = \dfrac{t^4}{4} + t^2 + 3t \end{cases}$$

$$\int uv'dt = uv - \int u'v\,dt$$

$$V(t) = \left(\frac{t^4}{4} + t^2 + 3t\right)\ln t - \int \underbrace{\frac{1}{t}\left(\frac{t^4}{4} + t^2 + 3t\right)}_{\frac{t^3}{4} + t + 3} dt$$

$$V(t) = \left(\frac{t^4}{4} + t^2 + 3t\right)\ln t - \frac{t^4}{16} + \frac{t^2}{2} + 3t + cte$$