Page 21 du polycopié

4) Exercices

Exercice 1 : Calculer au moyen d'un passage en coordonnées polaires, les intégrales suivantes :

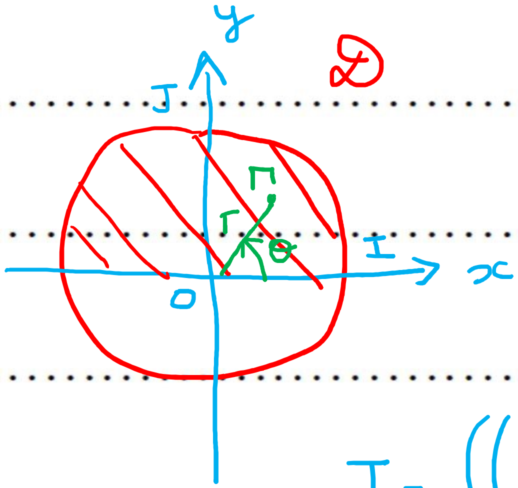
$$I=\iint_D \sqrt{x^2+y^2}\,dxdy \quad \text{où } D=\left\{(x,y)\,/\,x^2+y^2\leq 1\,;\ y\geq 0\right\}.$$

$$J=\iint_D \frac{dxdy}{(x^2+y^2)^2} \quad \text{avec } D=\left\{(x,y)\,/\ x\geq 1\,,\ x^2+y^2-2x\leq 0\right\}$$

Exercice 2 : Au moyen du changement de variable $\begin{cases} x+y=u \\ y=uv \end{cases}$ vérifier que :

$$\iint_D e^{\frac{y}{x+y}}\,dxdy=\frac{e-1}{2} \text{ lorsque } D=\left\{(x,y)\,/\,x\geq 0,\ y\geq 0,\ x+y\leq 1\right\}.$$

$$I = \iint_D \sqrt{x^2 + y^2}\, dxdy \quad \text{où } D = \left\{ (x,y) \,/\, x^2 + y^2 \le 1 \,;\, y \ge 0 \right\}$$



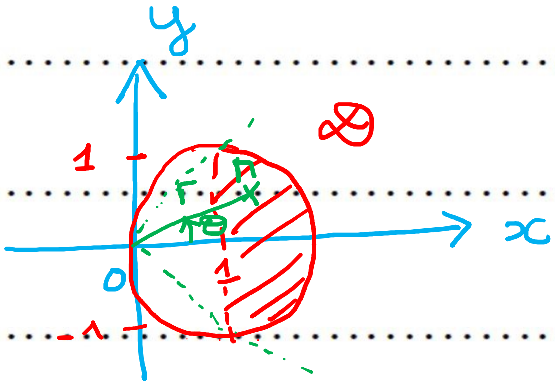
$\mathcal{D} \longrightarrow \Delta$

$$\begin{cases} 0 \le r \le 1 \\ 0 \le \theta \le \pi \end{cases} \quad \text{donc } \Delta = [0;1] \times [0;\pi]$$

$$I = \iint_\Delta f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$

$$I = \iint_{[0;1]\times[0;\pi]} \sqrt{r^2\cos^2\theta + r^2\sin^2\theta}\; r\, dr\, d\theta = \iint_\Delta r^2\, dr\, d\theta = \int_0^1 r^2\, dr \times \int_0^\pi d\theta = \frac{\pi}{3}$$

$$J=\iint_D \frac{dxdy}{(x^2+y^2)^2} \quad \text{avec } D=\{(x,y)\,/\; x\geq 1\,,\; \underbrace{x^2+y^2-2x\leq 0}\}$$



$\underbrace{x^2-2x} + y^2 \leq 0$

$(x-1)^2-1+y^2 \leq 0$

$(x-1)^2+y^2 \leq 1 \leftarrow$ disque de centre Ω et de rayon 1

$\Omega \Downarrow (1;0)$

Méth 1 pour déterminer Δ : Avec les équations de D

$D = \{(x,y) \,/\, x \geq 1 \,;\, x^2+y^2-2x \leq 0\}$

$\Delta = \{(r,\theta) \,/\, r\cos\theta \geq 1 \,;\, r^2-2r\cos\theta \leq 0\}$

$\underbrace{r}_{>0}(r-2\cos\theta) \leq 0$

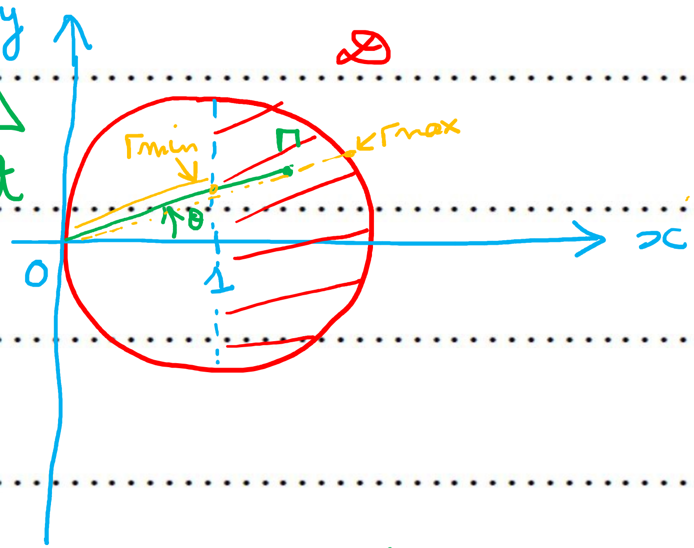
$r \leq 2\cos\theta$

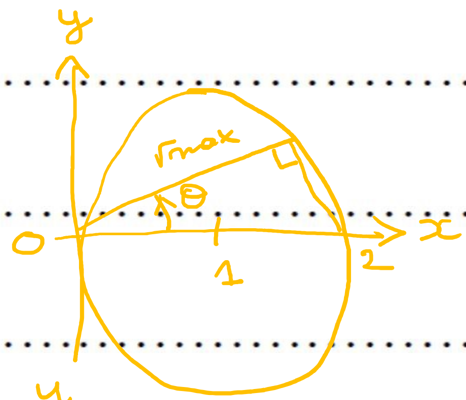
Sur le graphe : $-\frac{\pi}{4} \leq \theta \leq \frac{\pi}{4}$ donc $\cos\theta > 0$ et $r\cos\theta \geq 1 \iff r \geq \frac{1}{\cos\theta}$

$$\left\{\begin{array}{l} -\frac{\pi}{4} \leq \theta \leq \frac{\pi}{4} \\ \frac{1}{\cos\theta} \leq r \leq 2\cos\theta \end{array}\right.$$

Ré'th 2

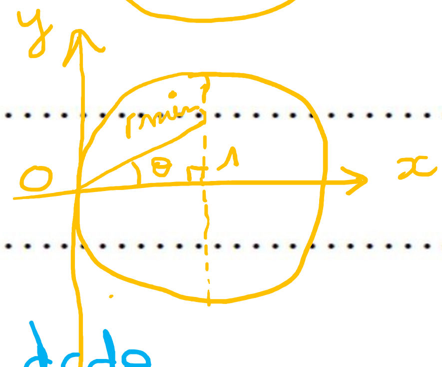
recherche de Δ graphiquement





$\cos\theta = \dfrac{r_{max}}{2}$

$r_{max} = 2\cos\theta$



$\cos\theta = \dfrac{1}{r_{min}}$

$r_{min} = \dfrac{1}{\cos\theta}$

$$\Delta = \left\{ (r,\theta) \mid -\frac{\pi}{4} \leq \theta \leq \frac{\pi}{4} \; ; \; \frac{1}{\cos\theta} \leq r \leq 2\cos\theta \right\}$$

$$I = \iint_{\mathcal{D}} \frac{dx\,dy}{(x^2+y^2)^2} = \iint_{\Delta} \frac{r\,dr\,d\theta}{r^4} = \iint_{\Delta} \frac{1}{r^3}\,dr\,d\theta$$

$$I = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \int_{\frac{1}{\cos\theta}}^{2\cos\theta} \frac{1}{r^3}\,dr\;d\theta = \int_{-\pi/4}^{\pi/4} \left[-\frac{1}{2r^2} \right]_{\frac{1}{\cos\theta}}^{2\cos\theta} = \frac{1}{2}\int_{-\pi/4}^{\pi/4} \left(\frac{-1}{4\cos^2\theta} + \cos^2\theta \right) d\theta$$

$$I = \frac{1}{2}\int_{-\pi/4}^{\pi/4}\left(-\frac{1}{4\cos^2\theta} + \cos^2\theta\right) d\theta = \int_0^{\pi/4}\left(-\frac{1}{4\cos^2\theta} + \cos^2\theta\right) d\theta$$

$$I = -\frac{1}{4}\left[\tan\theta\right]_0^{\pi/4} + \frac{1}{2}\int_0^{\pi/4}\left(1+\cos(2\theta)\right) d\theta$$

$$= -\frac{1}{4}(1-0) + \frac{1}{2}\left[\theta + \frac{\sin(2\theta)}{2}\right]_0^{\pi/4}$$

$$= -\frac{1}{4} + \frac{1}{2}\left(\frac{\pi}{4} + \frac{1}{2}\right)$$

$$I = \frac{\pi}{8}$$

<u>Exercice 2</u> : Au moyen du changement de variable $\begin{cases} x+y=u \\ y=uv \end{cases}$ vérifier que : ....

$$\iint_D e^{\frac{y}{x+y}}\,dxdy = \frac{e-1}{2} \text{ lorsque } D=\{(x,y)\,/\,x\geq 0,\ y\geq 0,\ x+y\leq 1\}.$$ ......

$$\frac{D(x,y)}{D(u,v)} = \begin{vmatrix} x'_u & x'_v \\ y'_u & y'_v \end{vmatrix}$$ et $\Delta = \{(u,v)\,/\ \dots$

$$\begin{cases} x = u - y \\ y = u.v \end{cases} \Rightarrow \begin{cases} x = u - uv \\ y = u.v \end{cases}$$

$$\frac{D(x,y)}{D(u,v)} = \begin{vmatrix} 1-v & -u \\ v & u \end{vmatrix} = (1-v)u + uv = u$$

$\Delta = \{(u,v)\,/\,u(1-v)\geq 0;\ u.v\geq 0;\ u\leq 1\}$

$u\geq 0$: $1-v\geq 0$, $v\leq 1$

$u\leq 0$: $1-v\leq 0$, $v\geq 1$

$$\Delta = \left\{ (u,v) \,/\, 0 \leq u \leq 1 \text{ et } 0 \leq v \leq 1 \right\} = [0;1]^2$$

$$\frac{D(x,y)}{D(u,v)} = u \qquad \begin{cases} x = u - uv \\ y = uv \end{cases}$$

$$I = \iint_{\mathcal{D}} e^{\frac{y}{x+y}}\, dx\, dy = \iint_{[0;1]^2} e^{\frac{uv}{u}} \, |u| \, du\, dv$$

$$\iint_{\mathcal{D}} f(x,y)\, dx\, dy = \iint_{\Delta} f\big(x(u,v), y(u,v)\big) \times \left| \frac{D(x,y)}{D(u,v)} \right| du\, dv$$

$$I = \iint_{[0;1]^2} e^{v} \cdot u \, du\, dv = \int_0^1 e^{v}\, dv \times \int_0^1 u\, du = \left[e^{v} \right]_0^1 \times \left[\frac{u^2}{2} \right]_0^1 = \frac{e-1}{2}$$

Exos Sup.

Calculer $I = \iint_D (1 + 4xye^{x^2}e^{y^2})\,dxdy$ où $D = [0;1] \times [0;2]$

$$I = \int_0^1 \int_0^2 (1+4xye^{x^2}e^{y^2})\,dy\,dx = \int_0^1 \left[y + 2xe^{x^2}e^{y^2}\right]_0^2 dx$$

$$I = \int_0^1 \left(2 + \underbrace{2xe^{x^2}e^4 - 2xe^{x^2}}_{2xe^{x^2}(e^4-1)}\right) dx = \left[2x + e^{x^2}(e^4-1)\right]_0^1$$

$$I = 2 + e(e^4-1) - (e^4-1) = 2 + (e-1)(e^4-1).$$

ou $I = 2 + e^5 - e - e^4 + 1 = 3 + e^5 - e - e^4$

Calculer $I = \iint_D y.\cos(xy)\,dxdy$ où $D = [0;\pi]^2$

$$I = \int_0^{\pi}\int_0^{\pi} y.\cos(xy)\,dx\,dy = \int_0^{\pi} \left[\sin(xy)\right]_0^{\pi} dy = \int_0$$

$$= \int_0^{\pi} \sin(\pi y)\,dy = \left[-\frac{\cos(\pi y)}{\pi}\right]_0^{\pi} = \frac{1-\cos(\pi^2)}{\pi}$$