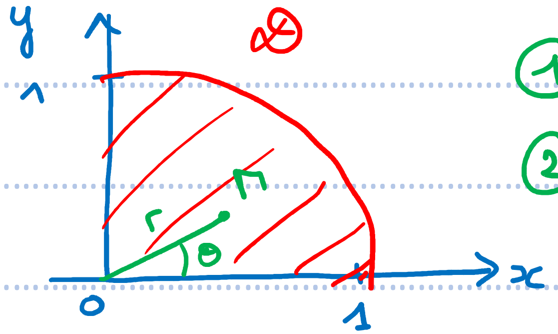
Corrigé I4 Exo2 et exo3 page 34 ChapIntDouble

Notes Chap 2 _ partie C : Intégrals doubles _ page 34 _ Exo 2

$$I_4 = \iint_{\mathcal{D}} \frac{y}{x^2+1}\,dx\,dy \quad \text{où} \quad \mathcal{D} = \left\{(x,y)\in\mathbb{R}^2 / \ x\geq 0 \ ; \ y\geq 0 \ ; \ \underbrace{x^2+y^2\leq 1}_{\text{disque de centre } O(0;0) \text{ et de rayon } 1.}\right\}$$



① Méthode 1 : passage en coordonnées polaires

② Méthode 2 : pas de changement de variable

① $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$ $\quad dx\,dy = r\,dr\,d\theta \quad \Delta = \left\{(r,\theta)\in\mathbb{R}^2 / \ 0\leq r\leq 1 \ ; \ 0\leq\theta\leq\frac{\pi}{2}\right\}$

$$I_4 = \iint_{\Delta} \frac{r\sin\theta}{(r\cos\theta)^2+1}\, r\,dr\,d\theta = \int_0^1\int_0^{\pi/2} \frac{r^2\sin\theta}{(r\cos\theta)^2+1}\,d\theta\,dr$$

$$\int \frac{U'}{1+U^2}\,d\theta = \operatorname{Arctan} U + cte \qquad \text{ici } U = r\cos\theta \Rightarrow U' = -r\sin\theta$$

Notes

Donc $I_4 = -\int_0^1 r . \int_0^{\pi/2} \frac{-r\sin\theta}{(r\cos\theta)^2+1}\,d\theta\,dr$

$$I_4 = -\int_0^1 r.\left[\arctan(r\cos\theta)\right]_0^{\pi/2} dr = -\int_0^1 r\Big(\underbrace{\arctan(r\underbrace{\cos\tfrac{\pi}{2}}_{})}_{0} - \arctan(r\underbrace{\cos 0}_{1})\Big)\,dr$$

$$I_4 = +\int_0^1 r.\arctan r\,dr.$$

IPP $\int_a^b uv'\,dr = [uv]_a^b - \int_a^b u'v\,dr.$

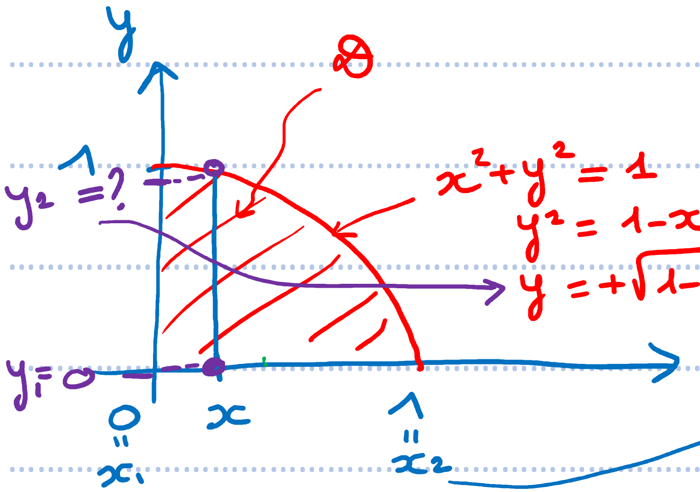
ALPES $\Rightarrow \begin{cases} u = \arctan r \\ v' = r \end{cases} \Rightarrow \begin{cases} u' = \frac{1}{1+r^2} \\ v = \frac{r^2}{2} \end{cases}$

ASTUCE

$$I_4 = \left[\frac{r^2}{2}.\arctan r\right]_0^1 - \frac{1}{2}\int_0^1 \frac{r^2+1-1}{\underbrace{1+r^2}_{1-\frac{1}{1+r^2}}}\,dr = \frac{1}{2}\arctan 1 - \frac{1}{2}.\int_0^1 1\,dr + \frac{1}{2}\int_0^1 \frac{1}{1+r^2}\,dr$$

$$I_4 = \frac{\pi}{8} - \frac{1}{2}[r]_0^1 + \frac{1}{2}[\arctan r]_0^1 = \frac{\pi}{8} - \frac{1}{2} + \frac{\pi}{8} = \frac{\pi}{4} - \frac{1}{2}.$$

② Méthode 2 $\quad I_4 = \iint_{D} \frac{y}{x^2+1}\,dx\,dy = \int_0^1 \int_0^{\sqrt{1-x^2}} \frac{y}{x^2+1}\,dy\,dx$



$x^2 + y^2 = 1$

$y^2 = 1 - x^2$

$y = +\sqrt{1-x^2}\ (>0)$

$$I_4 = \int_0^1 \frac{1}{x^2+1} \int_0^{\sqrt{1-x^2}} y\,dy\,dx$$

$$= \int_0^1 \frac{1}{1+x^2} \cdot \left[\frac{y^2}{2}\right]_0^{\sqrt{1-x^2}} dx$$

$$I_4 = \frac{1}{2} \int_0^1 \frac{1-x^2}{1+x^2}\,dx \qquad \int \frac{1}{1+x^2}\,dx = \arctan x + cte$$

Méthode a

$$\begin{array}{r|l} A = -x^2 + 1 & x^2 + 1 = B \\ \hline -(-x^2 - 1) & -1 = Q \\ \hline R = 2 & \end{array}$$

$$\frac{A}{B} = Q + \frac{R}{B} = -1 + \frac{2}{1+x^2}$$

Méthode b

$$\frac{1-x^2}{1+x^2} = \frac{1}{1+x^2} - \frac{\overbrace{x^2+1-1}^{\text{Astuce}}}{1+x^2} = \frac{1}{1+x^2} - 1 + \frac{1}{1+x^2}$$

$$I_4 = \frac{1}{2} \int_0^1 \left(-1 + \frac{2}{1+x^2}\right) dx = \frac{1}{2}\left\{ [-x]_0^1 + 2[\arctan x]_0^1 \right\}$$

$$I_4 = \frac{1}{2}\left(-1 + 2\pi/4\right) = \frac{\pi}{4} - \frac{1}{2}$$ ok

Notes

Chapitre 4 – page 34 – Exo 3

$(R>0)$

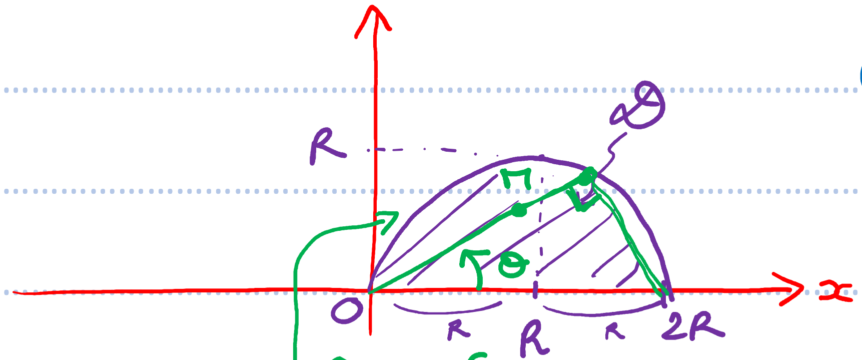
$I = \iint_{\mathcal{D}} x^2\,dx\,dy$ où $\mathcal{D}$ est le demi-disque de centre $A(R;0)$ et de rayon R tel que $y \geq 0$.

On passe en coordonnées polaires : $x = r\cos\theta$

$y = r\sin\theta$

$dx\,dy = r\,dr\,d\theta$

Jacobien calculé en cours.



$\cos\theta = \dfrac{r_{max}}{2R}$

Rappel : disque de centre $A(a;b)$ et de rayon R : $(x-a)^2+(y-b)^2 \leq R^2$

Nouveau domaine :

Méthode 1 graphique : $0 \leq \theta \leq \frac{\pi}{2}$ et $0 \leq r \leq r_{max}$?

$0 \leq r \leq 2R\cos\theta$

Méthode 2 inégalités : $\mathcal{D} = \{(x,y) / (x-R)^2 + (y-0)^2 \leq R^2$

$\Delta = \{(r,\theta) / (r\cos\theta - R)^2 + (r\sin\theta)^2 \leq R^2$

$\iff r^2\cos^2\theta - 2rR\cos\theta + \cancel{R^2} + r^2\sin^2\theta \leq \cancel{R^2} \iff r^2 - 2rR\cos\theta \leq 0$

Notes

$\iff \underbrace{r}_{>0}(r-2R\cos\theta) \leq 0 \iff r-2R\cos\theta \leq 0 \iff r \leq 2R\cos\theta.$

Donc $\Delta = \{(r,\theta) \; / \; 0 \leq r \leq 2R\cos\theta \; ; \; 0 \leq \theta \leq \frac{\pi}{2}\}$.

$$I = \iint_\Delta (r\cos\theta)^2 r\,dr\,d\theta = \int_0^{\pi/2}\int_0^{2R\cos\theta} r^3\cos^2\theta\,dr\,d\theta = \int_0^{\pi/2}\cos^2\theta \cdot \int_0^{2R\cos\theta} r^3\,dr\,d\theta$$

$$= \int_0^{\pi/2}\cos^2\theta \cdot \left[\frac{r^4}{4}\right]_0^{2R\cos\theta} d\theta = \frac{1}{4}\int_0^{\pi/2}\cos^2\theta \times 2^4R^4\cos^4\theta\,d\theta = \frac{16}{4}R^4 \cdot \int_0^{\pi/2}\cos^6\theta\,d\theta.$$

Sera donné en DS si besoin est.

$\cos^2 a = \frac{1+\cos(2a)}{2}$

$$\cos^6\theta = (\cos^2\theta)^3 = \left(\frac{1+\cos(2\theta)}{2}\right)^3 = \frac{1}{8}\left(1^3 + 3\cdot 1^2\cdot\cos(2\theta) + 3\cdot 1\cdot\cos^2(2\theta) + \underbrace{\cos^3(2\theta)}_{=\cos^2(2\theta)\times\cos(2\theta)}\right)$$

$\cos a \cos b = \frac{\cos(a+b)+\cos(a-b)}{2}$

$$= \frac{1}{8}\left(1+3\cos(2\theta)+3\cdot\frac{1+\cos(4\theta)}{2} + \frac{1+\cos(4\theta)}{2}\cdot\cos(2\theta)\right)$$

$$= \frac{1}{8}\left(1+3\cos(2\theta)+\frac{3}{2}+\frac{3}{2}\cos(4\theta)+\frac{1}{2}\cos(2\theta)+\frac{1}{2}\cos(\underset{b}{2\theta})\cos(\underset{a}{4\theta})\right)$$

$$= \frac{1}{8}\left(1+\frac{7}{2}\cos(2\theta)+\frac{3}{2}+\frac{3}{2}\cos(4\theta)+\frac{1}{4}\cos(6\theta)+\frac{1}{4}\cos(2\theta)\right)$$

Notes

$$\cos^6\theta = \frac{1}{8}\left(\frac{5}{2} + \frac{15}{4}\cos(2\theta) + \frac{3}{2}\cos(4\theta) + \frac{1}{4}\cos(6\theta)\right)$$

$$\cos^6\theta = \frac{1}{32}\left(10 + 15\cos(2\theta) + 6\cos(4\theta) + \cos(6\theta)\right)$$ ← Donné en DS, si besoin est.

$$I = \frac{4R^4}{32}\cdot\int_0^{\pi/2}\left(10 + 15\cos(2\theta) + 6\cos(4\theta) + \cos(6\theta)\right)d\theta.$$

$$= \frac{R^4}{8}\left[10\theta + \frac{15}{2}\sin(2\theta) + \frac{3}{2}\sin(4\theta) + \frac{1}{6}\sin(6\theta)\right]_0^{\pi/2}$$

$$= \frac{R^4}{8}\times 10\frac{\pi}{2}$$

$$I = \frac{5\pi R^4}{8}$$